ORIGINAL ARTICLE

A novel design of a sixth-order nonlinear modeling for solving engineering phenomena based on neuro intelligence algorithm

Zulqurnain Sabir[1] · Muhammad Asif Zahoor Raja[2] · Muhammad Shoaib[3] · R. Sadat[4] · Mohamed R. Ali[5,6]

Abstract

The current study aims to present a novel design of a sixth-order (SO) nonlinear Emden–Fowler nonlinear system (SO-NSEFM) along with its five types. The novel design of SO-NSEFM is achieved using the typical second-order Emden–Fowler system. The detail of the singularity and shape factors is presented for each type of the SO-NSEFM. Three different examples of each type of the designed SO-NSEFM will be solved using the supervised neural network (SNN) Levenberg–Marquardt backpropagation approach (LMBA), i.e., SNN–LMBA. A reference dataset using the spectral collocation scheme with the proposed SNN–LMBA will be established for the designed SO-NSEFM. The achieved approximate outcomes of the designed SO-NSEFM are accessible using the procedures of testing, verification, and training of the proposed neural networks to reduce the MSE. For the efficiency, correctness, and effectiveness of the proposed SNN-LMBA, the investigations are presented through the proportional performances of regression, MSE results, correlation and error histograms (EHs), and regression.

Keywords Sixth-order nonlinear Emden–Fowler model · Shape factors · Levenberg–Marquardt backpropagation · Spectral collocation scheme

1 Introduction

The singular systems got more importance in the last few decades due to their wider range of applications in different areas of science, technology, and engineering. The present study is related to the nonlinear Emden–Fowler (NEF) model that is singular, often considered stiff, complicated, and difficult to solve due to its harder nature. The researchers

always showed interest to solve the NEF model using different analytical and numerical approaches. The NEF model has many applications in fluid dynamics, population evolution, relativistic mechanics, the study of pattern creation, and chemical reactor system. The NEF model is a second-order singular model, mathematically written as [1–5]

$$\begin{cases} \dfrac{d^2u}{dx^2} + \dfrac{\varepsilon}{x}\dfrac{du}{dx} + h(x)z(u) = 0, & \varepsilon \geq 1 \\ u(0) = \xi, & \dfrac{du(0)}{dx} = 0. \end{cases} \quad (1)$$

The shape factor value is represented by ε. When $h(x) = 1$, the NEF model becomes the singular Lane–Emden model (SLE) system, mathematically given as

$$\begin{cases} \dfrac{d^2u}{dx^2} + \dfrac{\varepsilon}{x}\dfrac{du}{dx} + z(u) = 0, & \varepsilon \geq 1 \\ u(0) = \xi, & \dfrac{du(0)}{dx} = 0. \end{cases} \quad (2)$$

The SLE model is shown in Eq. (2) proposed by famous astrophysicists Lane and Emden. This famous historical model is used in the temperature variation modeling of a spherical gas cloud, radiative cooling, mathematical physics,

✉ Mohamed R. Ali
  mohamed.reda@bhit.bu.edu.eg

1   Department of Mathematics and Statistics, Hazara University, Mansehra, Pakistan

2   Future Technology Research Center, National Yunlin University of Science and Technology, 123 University Road, Section 3, Douliou, Yunlin 64002, Taiwan, ROC

3   Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock 43600, Pakistan

4   Department of Mathematics, Zagazig Faculty of Engineering, Zagazig University, Zagazig, Egypt

5   Faculty of Engineering and Technology, Future University, Cairo, Egypt

6   Department of Basic Science, Faculty of Engineering at Benha, Benha University, Banha 13512, Egypt

stellar configuration, polytropic star structure in astrophysics, the cluster's galaxies modeling, and self-gravitating gas clouds [6–9]. The function $z(u)$ appears in several forms in the SLE model and always provides different forms, and $z(u) = u^m$ is the most common and popular form that has attracted the researcher community. It is observed that for $m = 0$ and 1, the SLE equation is known as a linear equation; otherwise, it shows nonlinear behavior. The second kind of SLE model shows the isothermal gas sphere when $z(u) = e^u$. Furthermore, some other forms of $z(u)$ show the nonlinearity, such as $\cos u$, $\sin u$, $\cosh u$, and $\sinh u$, etc. The SLE model becomes the white-dwarf model by taking $z(u) = (u^2 - C)^{1.5}$ presented by Chandrasekhar [10]. The SLM is also implemented in physical sciences [11], dusty fluid models [12], density profile of gaseous star [13], reactions based on catalytic diffusion [14], electromagnetic theory [15], sublinear neutral term [16], classical/quantum mechanics [17], morphogenesis [18], oscillating magnetic systems [19], and isotropic continuous media [20].

The singular models are always considered stiff, grim, difficult, and challenging to handle because of the singular point. There are not so many analytic and numerical methods to solve these models. Some reported methods in the literature to handle such singular models are the Adomian decomposition approach presented by Shawagfeh et al. [21]. Romas et al. [22] established the series approach for the analytic solutions of the SLE model. Singh et al. [7] proposed a Haar wavelet collocation scheme for solving SLE model. Dizicheh et al. [23] proposed the Legendre wavelet spectral scheme for solving the SLE model. Saeed et al. [24] applied the Haar Adomian scheme for solving the nonlinear fractional SLE equation. Hashemi et al. [25] solve the SLE model using the reproduced kernel and group preserving schemes. Sabir et al. [26] designed a new third order singular functional differential model and solved by the differential transformation scheme. Bender et al. [27] presented a perturbative method to solve the singularity-based models. Nouh [28] presented the results of singular systems to solve the Pade approximation and power series schemes. The solutions of the singular models using the heuristic and swarm optimization-based techniques have been presented in these citations [29–34]. Ma [35, 36] finds N-soliton solutions and tests the Hirota N-soliton conditions using the Hirota bilinear formulation. Ma et al. [37] depend on the Hirota bilinear form, the solution of quadratic function is driven for the Hirota bilinear equation, and then, they used symbolic computation to construct the lump waves. The characteristic properties of the given lump waves are detected. The motive of this study is to investigate a novel design of the sixth-order (SO) nonlinear system of Emden–Fowler equations. SO nonlinear system of Emden–Fowler model (NSEFM), i.e., SO-NSEFM are presented and numerically investigated using the supervised neural network (SNN)

Levenberg–Marquardt backpropagation approach (LMBA), i.e., SNN-LMBA. For more than two variables, the system model using the ordinary differential equations has gotten huge significance due to its wider range of applications in the scientific and engineering applications, e.g., chemical reactor, network flow in the biological field, astrophysics, nonlinear circuits, fluid dynamics, boundary layer theory, and control theory optimization [35–46]. The novelty of the current work in the light of above stated literature is presented as

- A novel SO-NSEFM is designed based on typical and standard form of Emden–Fowler system.
- The designed sixth-order Emden–Fowler model has been numerically treated using the strength of SNN-LMBA.
- A reference dataset using the spectral collocation scheme (SCS) with the proposed SNN-LMBA will be established for the novel SO-NSEFM.
- The matching of the outcomes establishes the worth of the proposed SNN-LMBA for solving the SNN-LMBA.
- The performance of the scheme through comparative studies on correlation, error histograms (EHs), mean square error (MSE), and regression metrics provide the proposed SNN-LMBA.

The remaining paper parts are given as: The construction of the novel SO-NSEFM together is provided in Sect. 2. The novel SO-NSEFM based examples are provided in Sect. 3. The detail of the proposed SNN-LMBA, essential explanation, and solutions of the novel SO-NSEFM through SNN-LMBA is derived in Sect. 4. The final declarations and future research reports are reported in last section.

2 Construction of the novel SO-NSEFM

In this section, the structure of the novel SO-NSEFM together with the shape factors and the singularity is provided. The initial conditions (ICs) of the novel SO-NSEFM are obtained using the typical NEF model. For the derivation of the novel SO-NSEFM, the mathematical formulation is given as

$$
\begin{cases}
x^{-p_1} \dfrac{d^{a_1}}{dx^{a_1}} \left(x^{p_1} \dfrac{d^{a_2}}{dx^{a_2}} \right) u + h_1(x) f_1(u, v) = p(x), \\[2mm]
x^{-p_2} \dfrac{d^{a_1}}{dx^{a_1}} \left(x^{p_2} \dfrac{d^{a_2}}{dx^{a_2}} \right) v + h_2(x) f_2(u, v) = q(x),
\end{cases}
\tag{3}
$$

where p_1 and p_2 are taken as real and positive, $h_1(x)$ and $h_2(x)$ are the given function values, $p(x)$ and $q(x)$ are the forcing functions, and $f_1(u, v)$ and $f_2(u, v)$ known as the linear/nonlinear based functions of u and v. To design the novel SO-NSEFM, a_1 and a_2 must be taken as

$$a_1 + a_2 = 6, \quad a_1, \ a_2 \geq 1. \tag{4}$$

To satisfy the above equation, the following values are taken as:

$$a_1 = 5, \quad a_2 = 1, \tag{5}$$

$$a_1 = 4, \quad a_2 = 2, \tag{6}$$

$$a_1 = 3, \quad a_2 = 3, \tag{7}$$

$$a_1 = 2, \quad a_2 = 4, \tag{8}$$

$$a_1 = 1, \quad a_2 = 5. \tag{9}$$

The first type of the novel SO-NSEFM is obtained using the values of Eq. (5) in the system (3) and given as

$$\begin{cases} x^{-p_1} \dfrac{d^5}{dx^5}\left(x^{p_1}\dfrac{d}{dx}\right)u + h_1(x)f_1(u,v) = p(x), \\ x^{-p_2} \dfrac{d^5}{dx^5}\left(x^{p_2}\dfrac{d}{dx}\right)v + h_2(x)f_2(u,v) = q(x), \end{cases} \tag{10}$$

The simplified form of the derivatives in Eq. (10) is given as

$$\begin{cases} u(0) = I_1, \quad \dfrac{du(0)}{dx} = \dfrac{d^2u(0)}{dx^2} = \dfrac{d^3u(0)}{du^3} = \dfrac{d^4u(0)}{dx^4} = \dfrac{d^5u(0)}{dx^5} = 0, \\ v(0) = I_2, \quad \dfrac{dv(0)}{dx} = \dfrac{d^2v(0)}{dx^2} = \dfrac{d^3v(0)}{du^3} = \dfrac{d^4v(0)}{dx^4} = \dfrac{d^5v(0)}{dx^5} = 0. \end{cases} \tag{13}$$

The obtained model is given in Eqs. (12) and (13), which shows the multiple singularities, sixth-order form, nonlinearity, and a system of differential equations. The singular points at $x = 0$ appear five times for both the parameters u and v, respectively. The values of shape factor are $5p_1, 10p_1(p_1 - 1)$, $10p_1(p_1 - 1)(p_1 - 2)$, $5p_1(p_1 - 1)(p_1 - 2)(p_1 - 3)$ and $p_1(p_1 - 1)(p_1 - 2)(p_1 - 3)(p_1 - 4)$ for $u(x)$, whereas for the parameter $v(x)$, the shape factors are $5p_2, 10p_2(p_2 - 1)$, $10p_2(p_2 - 1)(p_2 - 2)$, $5p_2(p_2 - 1)(p_2 - 2)(p_2 - 3)$ and $p_2(p_2 - 1)(p_2 - 2)(p_2 - 3)(p_2 - 4)$, respectively. It is also observed that for $p_1 = p_2 = 1$, the 3rd, 4th, 5th, and 6th expressions vanish and the shape factor reduces to 5. For $p_1 = p_2 = 2$, the 4th, 5th, and 6th expressions vanish and shape factor reduces to 10 and 20, respectively. For $p_1 = p_2 = 3$, the 5th and 6th expressions vanish and the shape factor reduces to 15, 60, and 60, respectively. Likewise, for $p_1 = p_2 = 4$, the 6th expression vanishes and the shape factor reduces to 20, 120, 240, and 120, respectively.

$$\begin{cases} \dfrac{d^5}{dx^5}\left(x^{p_1}\dfrac{d}{dx}\right)u = x^{p_1}\dfrac{d^6u}{dx^6} + 5p_1 x^{p_1-1}\dfrac{d^5u}{dx^5} + 10p_1(p_1-1)x^{p_1-2}\dfrac{d^4u}{dx^4} + 10p_1(p_1-1)(p_1-2)x^{p_1-3}\dfrac{d^3u}{dx^3} \\ \qquad + 5p_1(p_1-1)(p_1-2)(p_1-3)x^{p_1-4}\dfrac{d^2u}{dx^2} + p_1(p_1-1)(p_1-2)(p_1-3)(p_1-4)x^{p_1-5}\dfrac{du}{dx}, \\ \dfrac{d^5}{dx^5}\left(x^{p_2}\dfrac{d}{dx}\right)v = x^{p_2}\dfrac{d^6v}{dx^6} + 5p_2 x^{p_2-1}\dfrac{d^5v}{dx^5} + 10p_2(p_2-1)x^{p_2-2}\dfrac{d^4v}{dx^4} + 10p_2(p_2-1)(p_2-2)x^{p_2-3}\dfrac{d^3v}{dx^3} \\ \qquad + 5p_2(p_2-1)(p_2-2)(p_2-3)x^{p_2-4}\dfrac{d^2v}{dx^2} + p_2(p_2-1)(p_2-2)(p_2-3)(p_2-4)x^{p_2-5}\dfrac{dv}{dx}, \end{cases} \tag{11}$$

The updated form of the model (11) using the above equation is written as

$$\begin{cases} \dfrac{d^6u}{dx^6} + \dfrac{5p_1}{x}\dfrac{d^5u}{dx^5} + \dfrac{10p_1(p_1-1)}{x^2}\dfrac{d^4u}{dx^4} + \dfrac{10p_1(p_1-1)(p_1-2)}{x^3}\dfrac{d^3u}{dx^3} + \dfrac{5p_1(p_1-1)(p_1-2)(p_1-3)}{x^4}\dfrac{d^2u}{dx^2} + \\ \qquad + \dfrac{p_1(p_1-1)(p_1-2)(p_1-3)(p_1-4)}{x^5}\dfrac{du}{dx} + h_1(x)f_1(u,v) = p(x), \\ \dfrac{d^6v}{dx^6} + \dfrac{5p_2}{x}\dfrac{d^5v}{dx^5} + \dfrac{10p_2(p_2-1)}{x^2}\dfrac{d^4v}{dx^4} + \dfrac{10p_2(p_2-1)(p_2-2)}{x^3}\dfrac{d^3v}{dx^3} + \dfrac{5p_2(p_2-1)(p_2-2)(p_2-3)}{x^4}\dfrac{d^2v}{dx^2} \\ \qquad + \dfrac{p_2(p_2-1)(p_2-2)(p_2-3)(p_2-4)}{x^5}\dfrac{dv}{dx} + h_2(x)f_2(u,v) = q(x). \end{cases} \tag{12}$$

The set of the ICs of the above Eq. (12) is given as

1. Methodology

Reference solutions
Designed dataset using the reference spectral collocation scheme to solve the novel SO-NSEFM

Intelligent computing
Multi-layer construction of the SNN-LMBA

Neuron Model

2. Results Simulations

Comparison of the results through SNN-LMBA with the solutions of Reference SCS together with the AE for all the examples of the novel SO-NSEFM

Result Comparison

AE

Proposed results through SNN-LMBA with MSE, EHs, State transition, Fitness and Regression to solve the novel SO-NSEFM

Performance

EHS

Fig. 1 Workflow diagram using the SNN-LMBA to solve the novel SO-NSEFM

Fig. 2 Designed system based on the single neuron
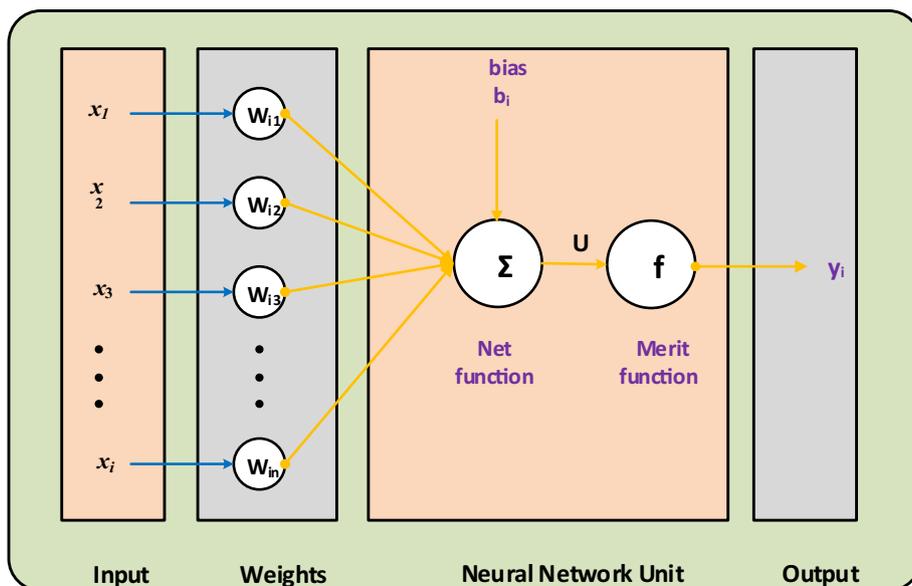
3 Methodology: SNN-LMBA

Fig. 3 Designed structure of the SNN-LMBA to solve the novel SO-NSEFM

The proposed scheme contains two steps: an essential explanation is provided to find the proposed SNN-LMBA dataset, while the execution process for the proposed SNN-LMBA is described (Fig. 1). The workflow diagram is represented in Fig. 2. The reference numerical results, i.e., datasets of SNN-LMBA are determined to execute the SCS. The proposed SNN-LMBA shows the multi-layer arrangements of SNNs along with LMBA. Figure 3 shows a single neuron system in SNNs scheme, while the proposed SNN-LMBA is executed using the routine of the 'nftool' of neural network in the MATLAB software together with the testing statics, suitably neurons setting, authentication data, learning methodology, and training data.

4 A designed SO-NSEFM and result simulations

In this section, an example of the novel SO-NSEFM is presented. The numerical results have been provided using the designed SNN-LMBA. The achieved numerical results using the SNN-LMBA are calculated in [0, 1] interval for the novel SO-NSEFM. The designed SNN-LMBA is implemented to solve the novel SO-NSEFM using 'nftool' routine in the Matlab software with ten hidden neurons, training data 80%, and testing/validation data are 10% using the optimization of SNN-LMBA. The designed SNN-LMBA is provided in Fig. 3, whereas the SNN-LMBA is accomplished to solve the novel SO-NSEFM.

(a) MSE for problem 1 of the novel SO-NSEFM

(b) MSE for problem 2 of the novel SO-NSEFM

(c) MSE for problem 3 of the novel SO-NSEFM
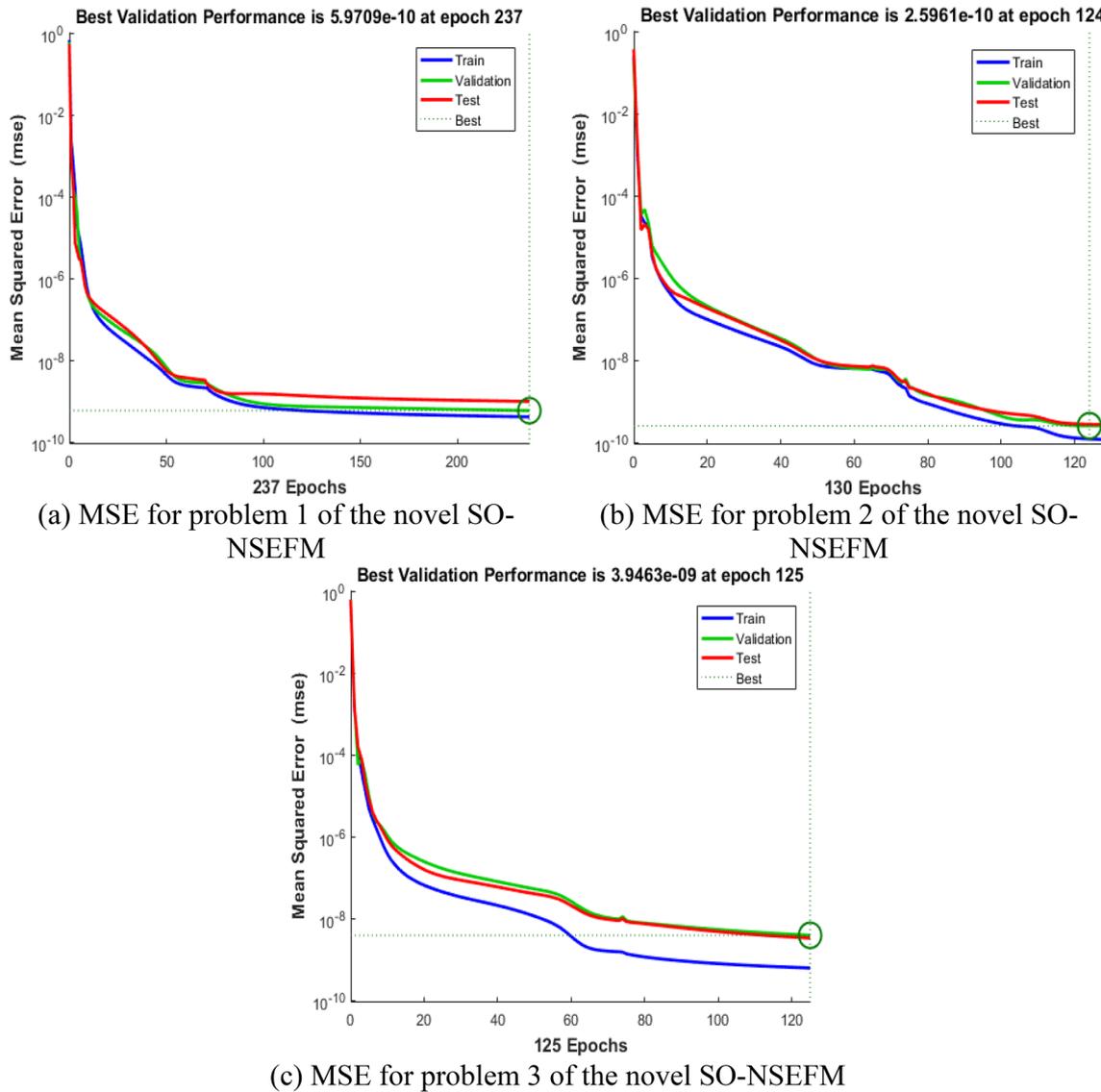
Fig. 4 Performance curves of the MSE values for the designed SNN-LMBA to solve the novel SO-NSEFM

4.1 SO-NSEFM

These designed model-based equations are obtained by taking the values of $p_1 = p_2 = 5$ in system (12).

Example 1: Consider

$$\begin{cases} \dfrac{d^6u}{dx^6} + \dfrac{25}{x}\dfrac{d^5u}{dx^5} + \dfrac{200}{x^2}\dfrac{d^4u}{dx^4} + \dfrac{600}{x^3}\dfrac{d^3u}{dx^3} + \dfrac{600}{x^4}\dfrac{d^2u}{dx^2} + \dfrac{120}{x^5}\dfrac{du}{dx} + u^2v = 181441 + x^6 - x^{12} - x^{18}, \\ \dfrac{d^6v}{dx^6} + \dfrac{25}{x}\dfrac{d^5v}{dx^5} + \dfrac{200}{x^2}\dfrac{d^4v}{dx^4} + \dfrac{600}{x^3}\dfrac{d^3v}{dx^3} + \dfrac{600}{x^4}\dfrac{d^2v}{dx^2} + \dfrac{120}{x^5}\dfrac{dv}{dx} + uv^2 = 181441 - x^6 - x^{12} + x^{18}. \end{cases} \tag{14}$$

The ICs of the above model is given as

$$\begin{cases} u(0) = 1, \ \dfrac{du(0)}{dx} = \dfrac{d^2u(0)}{dx^2} = \dfrac{d^3u(0)}{du^3} = \dfrac{d^4u(0)}{dx^4} = \dfrac{d^5u(0)}{dx^5} = 0, \\ v(0) = 1, \ \dfrac{dv(0)}{dx} = \dfrac{d^2v(0)}{dx^2} = \dfrac{d^3v(0)}{du^3} = \dfrac{d^4v(0)}{dx^4} = \dfrac{d^5v(0)}{dx^5} = 0. \end{cases} \tag{15}$$
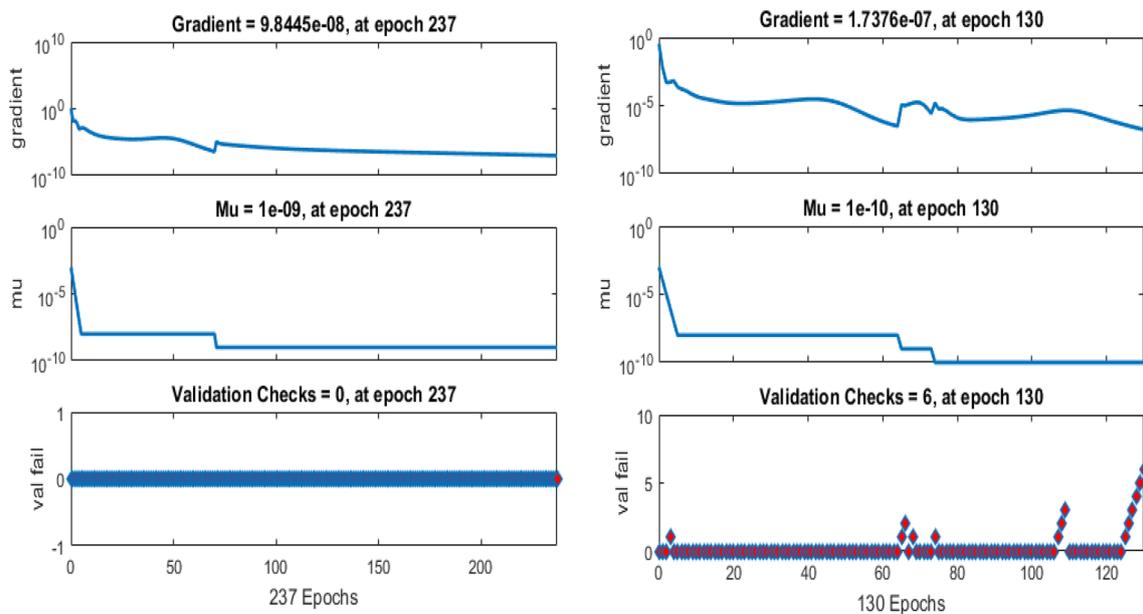
(a) State transition for problem 1 of the novel SO-NSEFM

(b) State transition for problem 2 of the novel SO-NSEFM
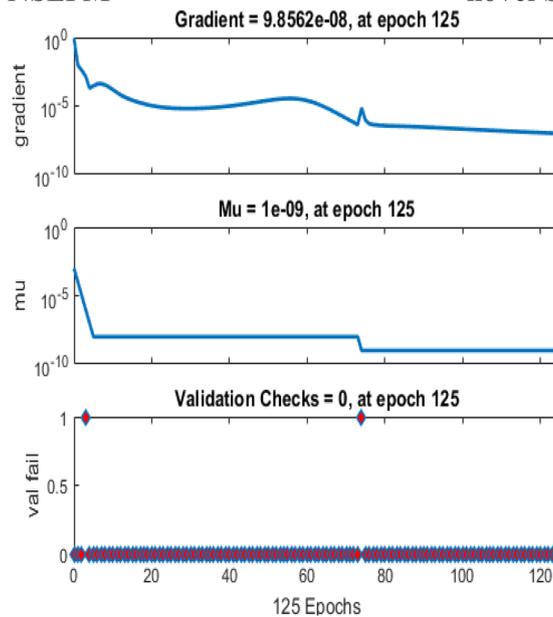
(c) State transition for problem 3 of the novel SO-NSEFM

Fig. 5 State transition curves for the designed SNN-LMBA to solve the novel SO-NSEFM

The plots of the example of the novel SO-NSEFM using the proposed SNN-LMBA are provided in Figs. 4, 5, 6, 7, 8, 9, 10, 11, 12. The results of the novel SO-NSEFM-based example are provided for the performance/transition states in Figs. 4, 5. The plots of the MSE convergence using the validation, training, best curve, and testing are given for all examples of the novel SO-NSEFM in Fig. 4. The best network performance is determined at epoch 237, 124, and 125 around 5.97×10^{-10}, 2.59×10^{-09}, and 3.94×10^{-09}, respectively. The gradient measures together are evaluated using the proposed SNN-LMBA for the example of novel SO-NSEFM are [9.84×10^{-08}, 1.37×10^{-07}, and 9.85×10^{-08}]

Fig. 6 Results comparison of the SNN-LMBA for Example 1 of the novel SO-NSEFM
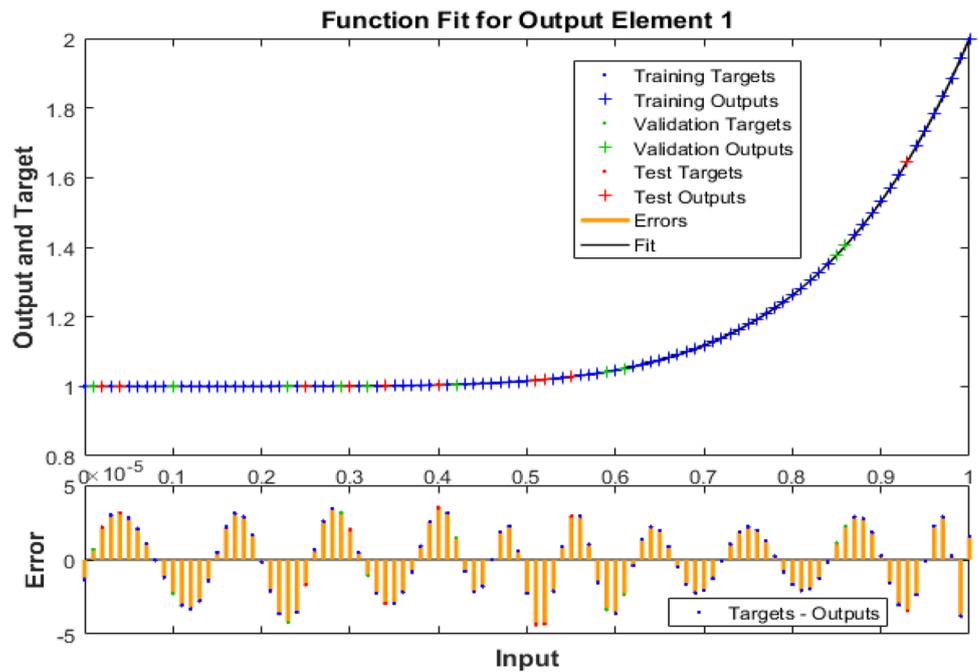
Fig. 7 Results comparison of the SNN-LMBA for Example 2 of the novel SO-NSEFM
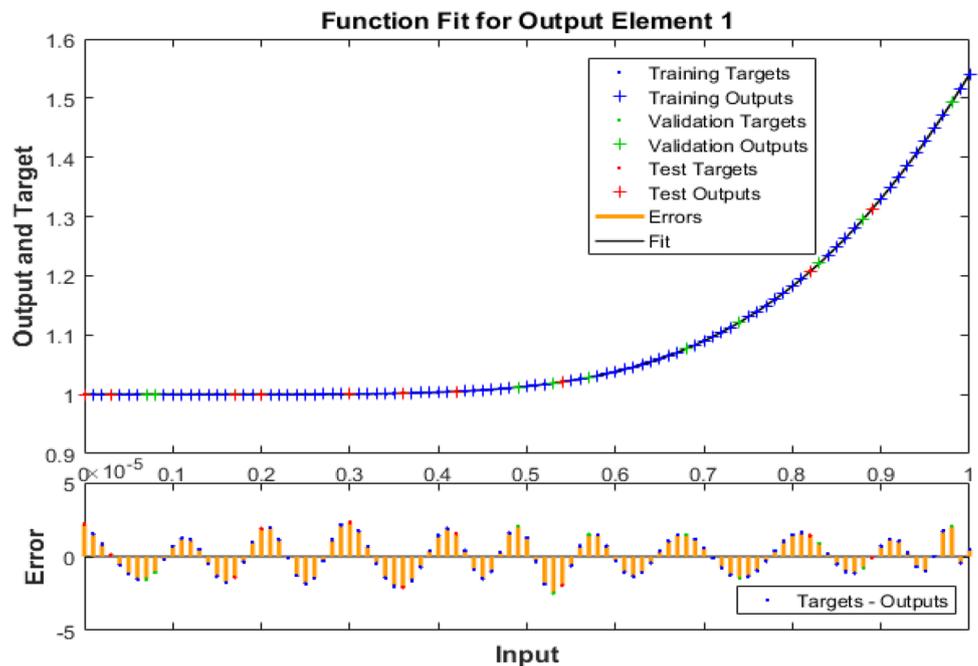
and $[1 \times 10^{-09}, 1 \times 10^{-10}, \text{and } 9.85 \times 10^{-08}]$ and presented in Fig. 5. These values indicate the correctness and convergence of the proposed SNN-LMBA of novel SO-NSEFM.

Figures 6, 7 and 8 show the curve fitting for the example of the novel SO-NSEFM. These figures specify the comparison of the SNN-LMBA with reference solutions of the novel SO-NSEFM together with the error plots for input span 0 to 1 using the values of the step size, i.e., 0.01. The values of maximum error for training, authentication, and testing-based proposed SNN-LMBA lie around 10^{-04} to 10^{-06} for all the example of the novel SO-NSEFM. The EH plots are presented in Fig. 9 that are used to examine the

Fig. 8 Results comparison of the SNN-LMBA for Example 3 of the novel SO-NSEFM
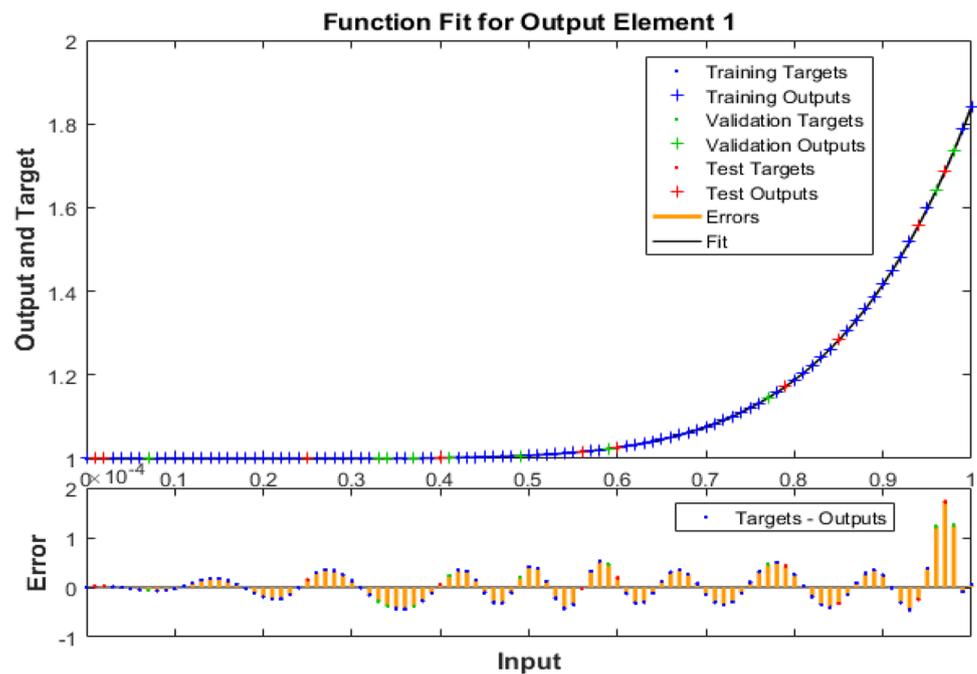
error investigation for the input/output intervals to solve each example of the novel SO-NSEFM. The EHs' values with zero-line reference have error lie around 2.18×10^{-06}, 1.24×10^{-06}, and 8.6×10^{-06} for the example of the novel SO-NSEFM. The plots of regression are provided in Figs. 10, 11, 12 for the example of the novel SO-NSEFM. These investigations using the correlation are performed to perform the regression investigation. It is also determined that the correlation (R) is calculated 1, which indicates the form of perfect system. This behavior shows the correctness of the proposed SNN-LMBA to solve the novel SO-NSEFM. Additionally, MSE convergence is attained through testing, performance, backpropagation procedures, validation, training, executed epochs, and complexity of time are provided in Table 1 to solve the novel SO-NSEFM.

Figures 13 and 14 show the comparison of the results obtained by the designed SNN-LMBA to solve the example using the novel SO-NSEFM. The parameter $u(x)$ and $v(x)$ results of the novel SO-NSEFM are plotted in Fig. 13a, b. The overlapping of the outcomes shows the correctness and the excellence of the designed scheme. The absolute error (AE) plots for the novel SO-NSEFM are drawn in Fig. 14. These AE values for $u(x)$ and $v(x)$ are provided in Fig. 14a, b. It is observed that the AE values for both of the parameters for $u(x)$ and $v(x)$ lie around $[10^{-04}, 10^{-06}]$ for example 1, $[10^{-04}, 10^{-05}]$ for example 2, and $[10^{-05}, 10^{-07}]$ for example 3, respectively. These

matching of the outcomes established the worth of the designed SNN-LMBA.

5 Conclusion

The present study is related to design a novel sixth-order Emden–Fowler singular nonlinear system using the typical Emden–Fowler system. Three examples of the novel model are provided together with the details of shape factors and singularities. It is seen that the first four types involve multiple singularities and shape factors, and the novel model has been designed based on the nonlinearity, logarithmic functions, exponential and trigonometric functions. The solutions of the novel sixth-order Emden–Fowler singular nonlinear system have been presented using the supervised LMBNNA. The approximation data are applied 80% for training, 10% for both testing and validation using the optimization procedure with ten hidden neurons. For the perfection of the designed model and scheme, the matching plots obtained by the designed scheme with the reference dataset are provided along with the absolute error (AE) plots. To check the mean square error, the convergence-based values for best curve, testing, training, and validation are presented for the nonlinear sixth-order Emden–Fowler singular model. The values of the correlation are provided for the regression. The values of the gradient using SNN-LMBA is evaluated for the designed

(a) EHs for problem 1 of the novel SO-NSEFM

(b) EHs for problem 2 of the novel SO-NSEFM

(c) EHs for problem 3 of the novel SO-NSEFM
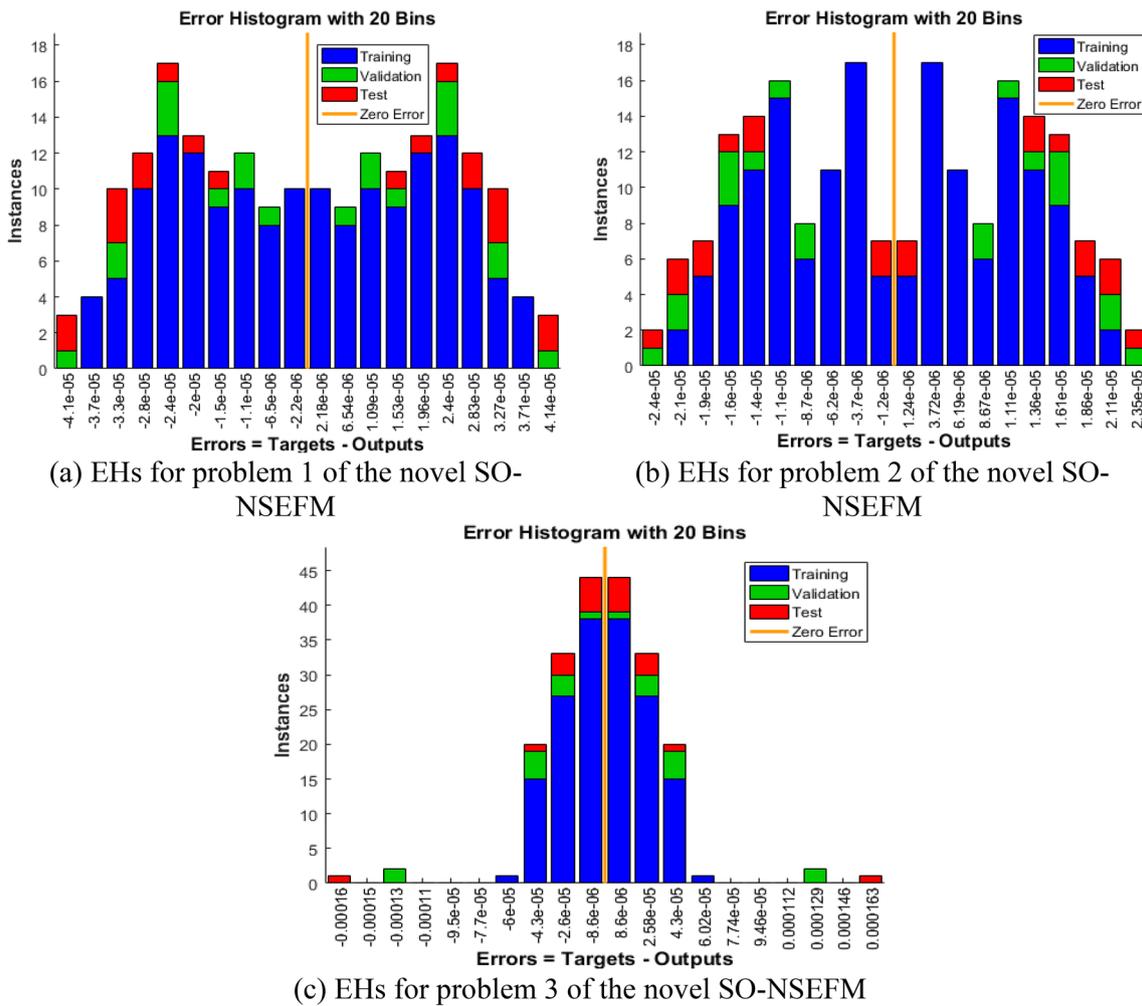
Fig. 9 EHs for the designed SNN-LMBA to solve the novel SO-NSEFM

nonlinear system. Moreover, the precision and accuracy are justified using the graphical as well as numerical illustrations of regression dynamics, error histograms, and convergence on MSE.

In future, a variety of singular, biological models, and fluid and fractional order models can be constructed and can be solved using the LMBNN [47–55].

Fig. 10 Regression performances for the designed SNN-LMBA to solve the Example 1 using the novel SO-NSEFM
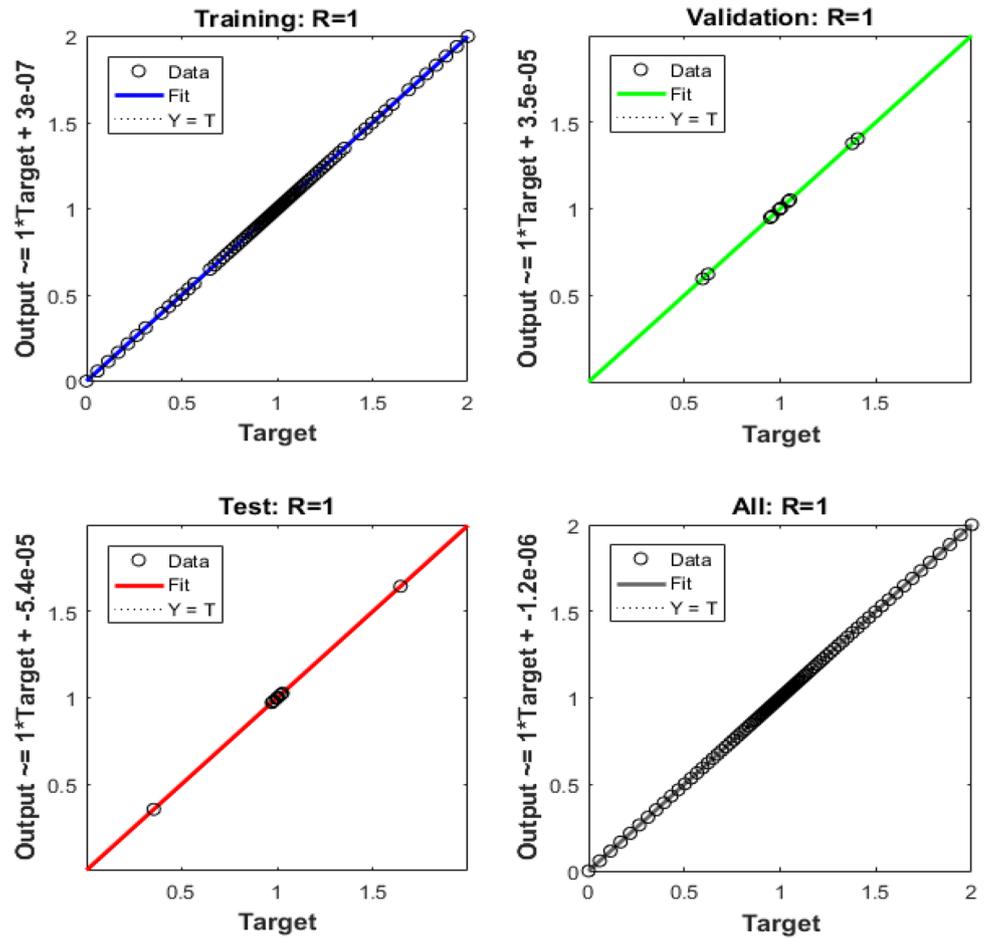
Fig. 11 Regression performances for the designed SNN-LMBA to solve the Example 2 using the novel SO-NSEFM
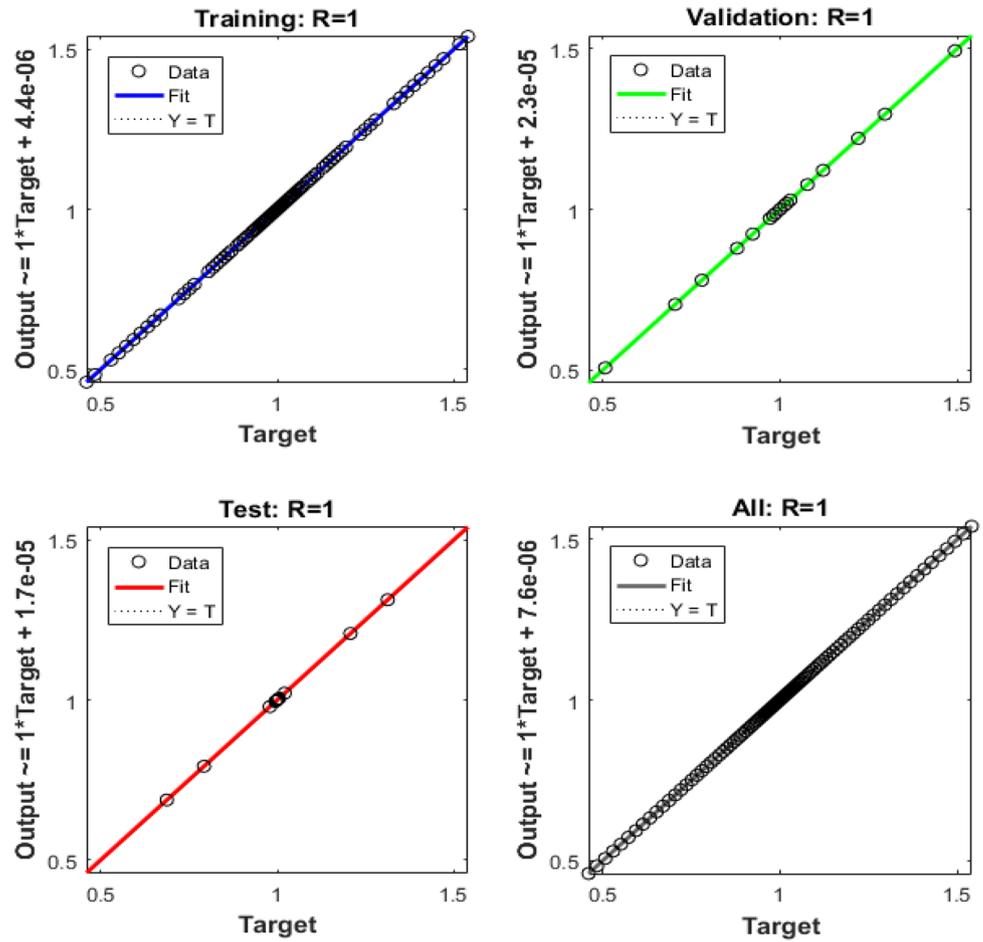
Fig. 12 Regression performances for the designed SNN-LMBA to solve the Example 3 using the novel SO-NSEFM
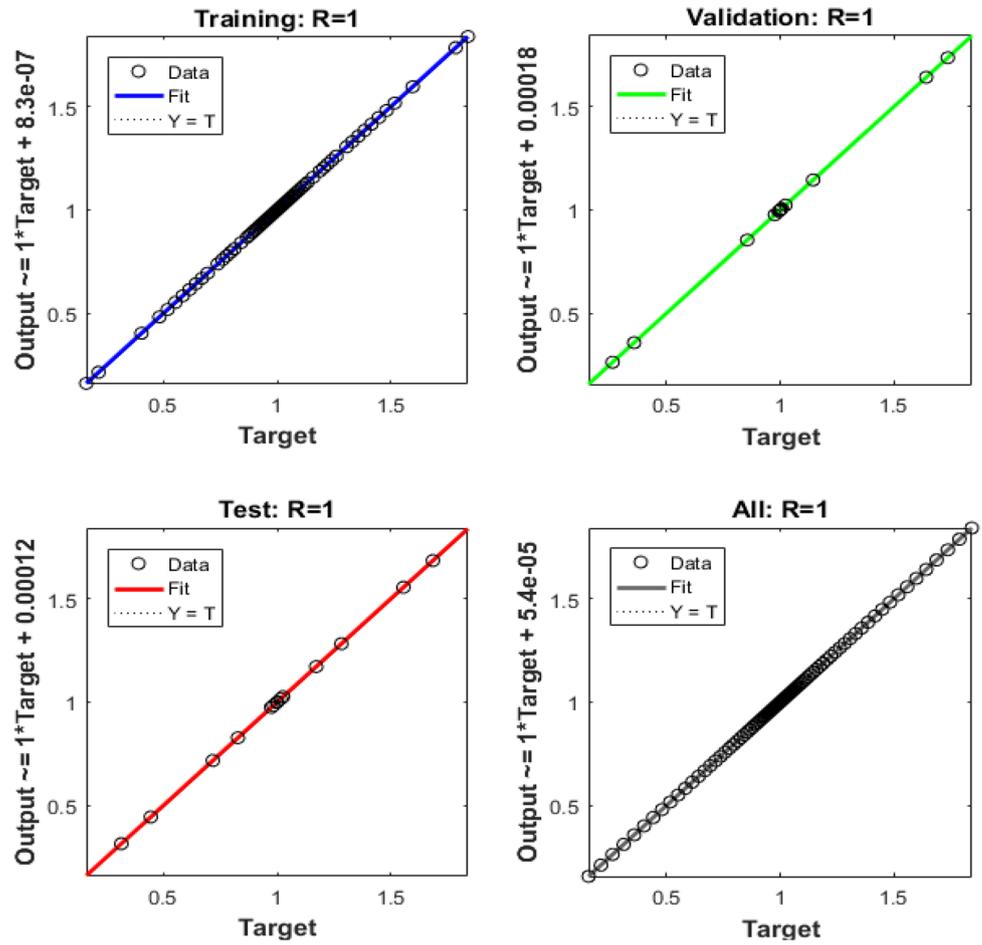
Table 1 SNN-LMBA results based on of the novel SO-NSEFM

Example	MSE			Performance	Gradient	Mu	Iteration	Time
	Training	Authentication	Testing					
1	4.19×10^{-10}	5.97×10^{-10}	1.00×10^{-09}	4.20×10^{-10}	9.84×10^{-08}	1.00×10^{-09}	237	2
2	1.24×10^{-10}	2.59×10^{-10}	2.82×10^{-10}	1.19×10^{-10}	1.74×10^{-07}	1.00×10^{-10}	130	1
3	6.28×10^{-10}	3.94×10^{-10}	8.92×10^{-10}	6.28×10^{-10}	9.86×10^{-08}	1.00×10^{-09}	125	1

(a) $u(x)$ results for Examples 1-3

(b) $v(x)$ results for Examples 1-3
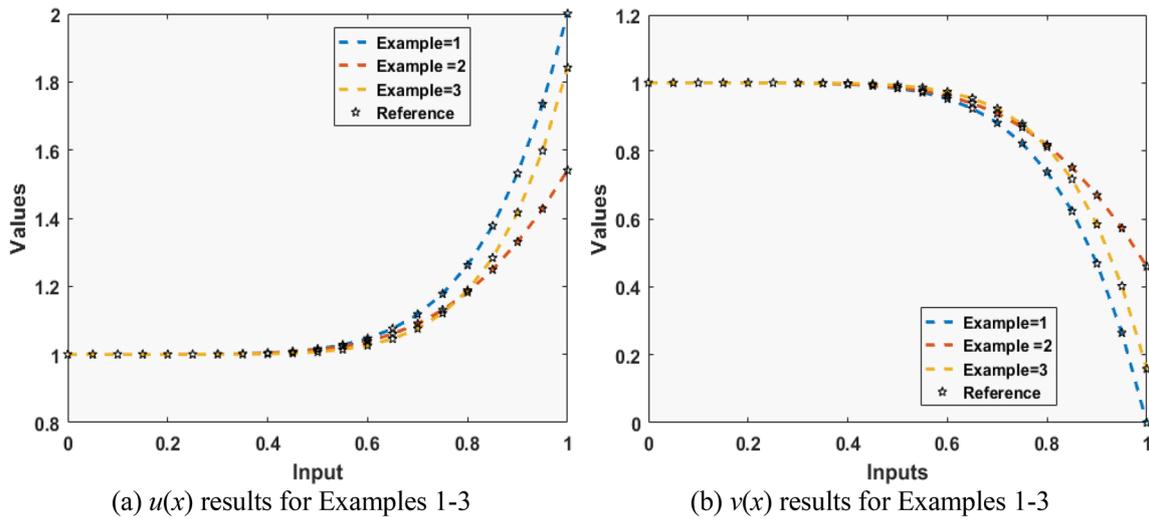
Fig. 13 Results' comparison of the designed SNN-LMBA to solve the Examples 1–3 using the novel SO-NSEFM

(c) AE values of $u(x)$ for Examples 1-3

(d) AE values of $v(x)$ for Examples 1-3
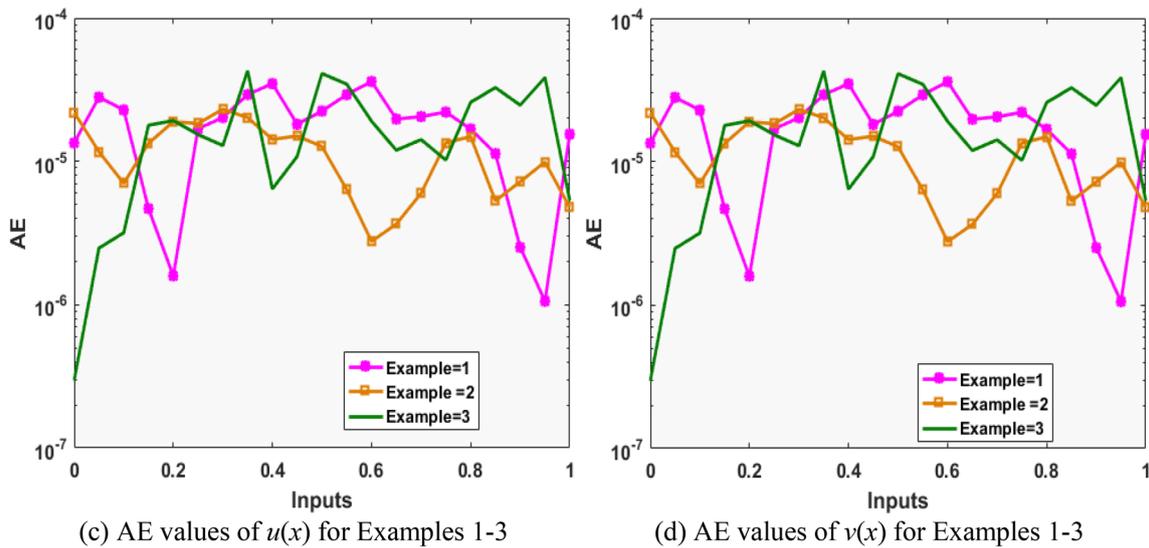
Fig. 14 Values of the AE of $u(x)$ and $v(x)$ to solve the Examples 1–3 using the novel SO-NSEFM

Declarations

Conflict of interest The authors state that they have no conflict of interest.

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